Moon shadow

The Moon shadow can be described using two angles, the clock angle, c, and the normal incidence angle, j. The "clock angle", c, indicates the angle between zenith and clockwise direction towards

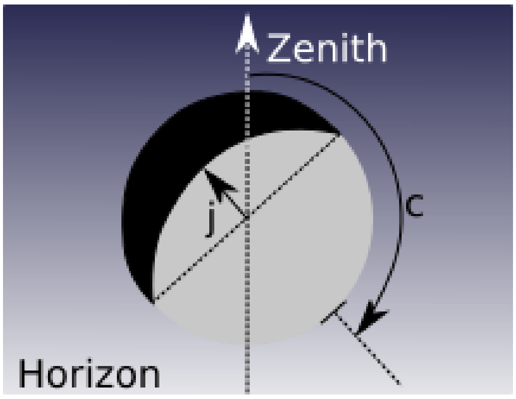

Figure 1: Upper limb elevation.

the sun, where 0 indicates Sun above Moon, 90 indicates Sun to the right etc. The "incidence angle", indicates angle between Moon and Sun. The normal incidence angle, j, is reported in degrees, so that -90 gives 100% shadow, 0 gives 50% and 90 gives 0% shadow.

Lunar illumination

The lunar illumination, b, is given by $b = \frac{1+\sin(j)}{2} \cdot 100\%$.

Approximating the clock angle

As the Moon moves over the sky, the apparent direction towards the Sun (clock angle) changes as the observer turns. The Moon shadow clock angle as a function of time can be approximated if you at one time know the Moon and Sun elevation, Moon clock angle and the position of the observer, assuming that Moon and Sun positions do not change. The algorithm for approximating the elevation of the Sun or Moon at a given time, given max and min elevations is given in another document. Note that zs is complementary to the Sun elevation, e_{Sun} so that $zs = \frac{\pi}{2} - e_{\mathrm{Sun}}$. Similar goes for the the Moon zenith angle, zm.

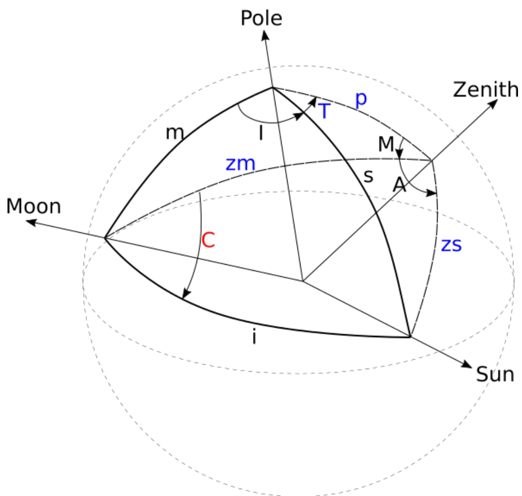

Figure 2: Spherical trigonometry

The known variables at a time, C, p, T, zs and zm are marked in red and blue, the varying arcs are dashed. We assume that m, s, i, p and I are constant, where s can be estimated from p

and the maximum Sun elevation, $e_{\max}$.

$$\begin{aligned} zs_{\min} &= \frac{\pi}{2} - e_{\max} \\ s &= p + zs_{\min}. \end{aligned}$$

Note that the Sun may have maximum elevation to the North at low latitudes (check the azimuth) in which case $s = p - zs_{\min}$. We may estimate m through a similar approach by using the maximum Moon elevation instead of the maximum Sun elevation. The angle I can be estimated by comparing the time difference between maximum Sun and Moon elevations and scaled so that a 6 hour difference matches an I angle of $\pi/2$. The time angle T is zero at noon and π at midnight.

Let us denote the angles at the later time with an asterix, for instance T^* is the time angle at the later time. Assuming that we know p, zs, zm and the clock angle C at time angle T, we would like to approximate the clock angle at a later time, C^*.

Adjusting s using p, zs and T (optional)

You may want to fine-adjust s so that it is consistent with p and zs at T. From figure 2, using the *cosine rule* from *spherical trigonometry* gives,

$$\cos(zs) = \cos(s)\cos(p) + \sin(s)\sin(p)\cos(T).$$

The problem here is that the limited information allows several geometrical solutions, a property that manifests itself through the appearence of both $\sin(s)$ and $\cos(s)$. To solve this equation for s, we use "Weierstrass t-substitution" and introduce t so that,

$$\begin{aligned} t &= \tan(\frac{s}{2}) \\ \sin(s) &= \frac{2t}{1+t^2} \\ \cos(s) &= \frac{1-t^2}{1+t^2} \end{aligned}$$

which gives us

$$\begin{aligned} \cos(zs) &= \frac{1-t^2}{1+t^2}\cos(p) + \frac{2t}{1+t^2}\sin(p)\cos(T) \\ t &= \frac{\sin(p)\cos(T) \pm \sqrt{\sin^2(p)\cos^2(T) - \cos^2(zs) + \cos^2(P)}}{(\cos(zs) + \cos(P))} = \tan(\frac{s}{2}) \end{aligned}$$

yielding several solutions for s as mentioned above. Pick the one closest to your original estimate. This correction procedure can also be repeated for zm.

Calculating i from m, s and I

From figure 2, using the *cosine rule* from *spherical trigonometry* gives,

$$\cos(i) = \cos(m)\cos(s) + \sin(m)\sin(s)\cos(I)$$

which can be solved for i.

Calculating $T*$

The new solar time angle is calculated by assuming that the earth spins around its axis once a day. The solar time angle is zero at noon, and π at midnight.

Calculating zs^* from p, s and T^*

From figure 2, using the *cosine rule* from *spherical trigonometry* gives,

$$\cos(zs^*) = \cos(s)\cos(p) + \sin(s)\sin(p)\cos(T^*)$$

which can be solved for zs^*.

Calculating zm^* from p, m, I and T^*

From figure 2, using the *cosine rule* from *spherical trigonometry* gives,

$$\cos(zm^*) = \cos(m)\cos(p) + \sin(m)\sin(p)\cos(I + T^*)$$

which can be solved for zm^*.

Calculating C^* from zm^*, i and zs^*

From figure 2, using the *cosine rule* from *spherical trigonometry* gives,

$$\begin{aligned}
\cos(zs^*) &= \cos(i)\cos(zm^*) + \sin(i)\sin(zm^*)\cos(C^*) \\
\cos(C^*) &= \frac{\cos(zs^*) - \cos(i)\cos(zm^*)}{\sin(i)\sin(zm^*)}
\end{aligned}$$

which can be solved for C^*. Note that the clock angle C^* is undefined if either i or zm is zero.